

The Topological Particle and Morse Theory

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Abstract

Canonical BRST quantization of the topological particle defined by a Morse function h is described. Stochastic calculus, using Brownian paths which implement the WKB method in a new way providing rigorous tunnelling results even in curved space, is used to give an explicit and simple expression for the matrix elements of the evolution operator for the BRST Hamiltonian. These matrix elements lead to a representation of the manifold cohomology in terms of critical points of h along lines developed by Witten [1].

1 Introduction

The topological particle, whose canonical BRST quantization is developed and applied in this paper, is the simplest example of a topological quantum theory. There are many reasons, both physical and mathematical, for studying such theories; however much of the work to date has been carried out in the Lagrangian approach, using the functional integral as the starting point. The underlying motivation for this paper is the belief that a serious canonical analysis of such theories should be fruitful.

The topological particle and its quantization is described by Beaulieu and Singer in [2], but these authors concentrate on the case based on a constant function on the manifold, while in this paper the model is based on a Morse function for the manifold, a function which is anything but constant, having only isolated critical points, and encodes information about the topology of the manifold. It is first shown, in section 2, that the supersymmetric quantum mechanical system which arises on BRST quantization of the model is the system used by Witten [1] in his work on supersymmetry and Morse theory. The topological origin of this model gives a natural

explanation for the form of the matrix elements for the theory which in Witten's paper are calculated by instanton methods.

The main new result of the present paper is a path integral formula for the calculation of these matrix elements, which is derived in section 4 using the methods of stochastic calculus on manifolds. Paths are defined by a stochastic differential equation which is essentially the Nicolai map for the model [3, 4]; the paths encode fluctuations about classical trajectories and thus lead to a fully rigorous path integral WKB method (as derived by Blau, Keski-Vakkuri and Niemi using physicists' methods [5]). The result makes contact again with the classical action of the topological particle which was the starting point.

Having established the precise form of the matrix elements of the model, in section 5 these are used to construct explicitly the cochains and cohomology for the model of the manifold cohomology introduced by Witten [1] based on the critical points of a Morse function.

2 Classical Dynamics

The topological particle model introduced by Beaulieu and Singer [2] is defined by the action

$$S(x(.)) = \int_0^t v_\mu(x(t')) \dot{x}^\mu(t') dt'. \quad (1)$$

The fields x are smooth maps $x : I \rightarrow M$ from I (the interval $[0, t]$ of the real line) into a compact n -dimensional Riemannian manifold M , while $v = dh$ is an exact one-form on M . The components v_μ in local coordinates $x^\mu, \mu = 1, \dots, n$ are as usual defined by $v = v_\mu dx^\mu$, so that $v_\mu = \frac{\partial h}{\partial x^\mu}$.

Clearly, since v is the differential of h , this action can be expressed more simply as

$$S(x(.)) = h(x(t)) - h(x(0)). \quad (2)$$

This form of the action shows that the model is indeed topological in nature, a related point being that the equation of motion for x is trivially satisfied. However the form of the action (1) involving positions and velocities is required for the passage from the Lagrangian to the Hamiltonian form. While Beaulieu and Singer considered the case $v = 0$, in this paper a more general situation is considered; in particular in section 5 the function h is taken to be a Morse function, that is, a function on M with isolated critical points.

It is evident that the action (1) is highly symmetric, depending only on the end-points of the path. It might thus be naively supposed that the path integral

$$\int_{\text{paths/symmetries}} \mathcal{D}x(.) \exp(S(x(.))) \quad (3)$$

would be trivial. This is not in fact the case because the ‘measure’ $\mathcal{D}x$ is not simply some limit of a product measure, but must be derived by careful canonical quantization of the theory, which is carried out below.

The first step in this process is to investigate the classical Hamiltonian dynamics. From the action (1) the Lagrangian of the theorem is seen to be

$$\mathcal{L}(x, \dot{x}) = v_\mu(x) \dot{x}^\mu, \quad (4)$$

so that the Euclidean time Legendre transformation to the phase space T^*M (the cotangent bundle of M) gives as momentum conjugate to x^μ

$$p_\mu = i \frac{\delta \mathcal{L}(x, \dot{x})}{\delta \dot{x}^\mu} = i v_\mu. \quad (5)$$

The symmetries of the system now manifest themselves as n constraints on the phase space T^*M :

$$T_\mu \equiv -p_\mu + i v_\mu(x) = 0, \quad \mu = 1, \dots, n. \quad (6)$$

The Poisson brackets on the phase space T^*M are obtained from the standard symplectic form $\omega = dp_\mu \wedge dx^\mu$, so that as usual $\{x^\mu, p_\nu\} = \delta_\nu^\mu$. Direct calculation shows that (since v is closed)

$$\{T_\mu, T_\nu\} = 0. \quad (7)$$

The Hamiltonian of the system is, by the Euclidean time prescription,

$$H = i p_\mu \dot{x}^\mu + \mathcal{L}(x, \dot{x}) = 0, \quad (8)$$

so that the constraints are first class and abelian.

As is standard in a topological theory, the constraints are of a number that seems to preclude any interesting dynamics - in this case the system has a $2n$ -dimensional phase space with n first class constraints, so that by naive counting one would expect the corresponding reduced phase space to be trivial. In fact the theory does capture some topological information as will emerge below.

The first indication of this comes from considering gauge-fixing, which shows that the reduced phase space, while as expected zero dimensional, corresponds to the

critical points of h . The reduced phase space is defined to be the quotient of the subspace of the phase space T^*M on which the constraints hold by the action of the group generated by the constraints [6, 7]. Classically gauge-fixing conditions are sought which pick out one point in each orbit of this group; in this case a natural choice is the set of n conditions $X^\mu \equiv g^{\mu\nu}(-p_\nu - iv_\nu) = 0$. (Justification for this choice can only be fully made on quantization.) Taken together the constraints and the gauge-fixing condition are satisfied when $p_\mu = 0$ and $v_\mu = 0$, that is, at the critical points of the manifold. To see how this finite reduced phase space can provide topological information we turn to quantization, using the BRST approach.

3 BRST quantization

To implement the constraints and gauge-fixing at the quantum level we use the BRST quantization in canonical form [8, 6], introducing ghosts and their conjugate momenta. For this process two supermanifolds are required, a super configuration space SM with even local coordinates x^μ and odd local coordinates η^μ and a super phase space SPM with even local coordinates x^μ and p_μ and odd local coordinates η^μ and π_μ . (In each case the index μ runs from 1 to n .) The (n, n) dimensional supermanifold SM is built from the tangent bundle of M , with coordinate patches corresponding to those on M and changes of the coordinates x^μ between patches being those on M while those of the coordinates η^μ are defined by

$$\tilde{\eta}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \eta^\nu. \quad (9)$$

For future reference we note that there is a well-defined projection $\epsilon : SM \rightarrow M$ defined by

$$\epsilon(x, \eta) = x. \quad (10)$$

The super phase space SPM is the cotangent bundle to SM , so that p_μ and π_μ transform according to the rule

$$\tilde{p}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} p_\nu, \quad \tilde{\pi}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \pi_\nu. \quad (11)$$

The simplest, and natural, choice of symplectic form on this manifold, which makes π_μ the conjugate momentum to η_μ , is

$$\begin{aligned} w_s &= d(p_\mu \wedge dx^\mu + \pi_\mu \wedge D\eta^\mu) \\ &= dp_\mu \wedge dx^\mu + D\pi_\mu \wedge D\eta^\mu - \frac{1}{2} R_{\mu\nu\lambda}{}^\kappa \pi_\kappa \eta^\lambda dx^\mu \wedge dx^\nu, \end{aligned} \quad (12)$$

where the Levi-Cevita connection corresponding to the Riemannian metric g has been used, with Christoffel symbols $\Gamma_{\mu\nu}{}^\kappa$ and curvature tensor components $R_{\mu\nu\lambda}{}^\rho$, so that

$$D\eta^\mu = d\eta^\mu + \Gamma_{\nu\lambda}{}^\mu \eta^\lambda dx^\nu, \quad D\pi_\mu = d\pi_\mu - \Gamma_{\nu\mu}{}^\lambda \pi_\lambda dx^\nu. \quad (13)$$

The corresponding Poisson brackets (which are calculated in appendix A) are:

$$\begin{aligned} \{p_\nu, x^\mu\} &= \delta_\nu^\mu, & \{p_\mu, p_\nu\} &= R_{\mu\nu\lambda}{}^\kappa \pi_\kappa \eta^\lambda \\ \{p_\mu, \eta^\nu\} &= \Gamma_{\mu\lambda}{}^\nu \eta^\lambda, & \{p_\mu, \pi_\lambda\} &= -\Gamma_{\mu\lambda}{}^\nu \pi_\nu, \\ \text{and } \{\pi_\nu, \eta^\mu\} &= \delta_\nu^\mu. \end{aligned} \quad (14)$$

the others being zero. To quantize, we take wave functions to be functions $\psi(x, \eta)$ on the super configuration space SM . The observables x^μ and η^μ are simply represented by multiplication by these variables, while the momenta p_μ and π_μ are represented as

$$p_\mu = -iD_\mu \equiv -i \left(\frac{\partial}{\partial x_\mu} + \eta^\nu \Gamma_{\mu\nu}{}^\lambda \frac{\partial}{\partial \eta^\lambda} \right) \quad \text{and} \quad \pi_\mu = -i \frac{\partial}{\partial \eta^\mu}. \quad (15)$$

The BRST operator Ω is constructed from the constraints in the standard way, giving

$$\Omega = \eta^\mu T_\mu = i\eta^\mu \left(\frac{\partial}{\partial x^\mu} + v_\mu \right). \quad (16)$$

(The symmetry of the Christoffel symmetry removes the covariant part of p_μ in this case, as in exterior differentiation of forms.) The gauge-fixing fermion χ is then constructed from the gauge-fixing functions X^μ in the standard way:

$$\chi = \pi_\mu x^\mu = ig^{\mu\nu} \pi_\mu (D_\nu - v_\nu). \quad (17)$$

States of the system can of course naturally be identified with forms on M via the identification

$$a_{\mu_1 \dots \mu_p}(x) \eta^{\mu_1} \dots \eta^{\mu_p} \leftrightarrow a_{\mu_1 \dots \mu_p}(x) dx^{\mu_1} \dots dx^{\mu_p} \quad (18)$$

Under this identification we see that

$$\begin{aligned} \Omega &= -i\eta^\mu \left(\frac{\partial}{\partial x_\mu} - v_\mu \right) = i(d + \eta^\mu v_\mu) = ie^h de^{-h}, \\ \chi &= -ig^{\mu\nu} \pi_\nu \left(\frac{\partial}{\partial x_\mu} + \eta^\nu \Gamma_{\mu\nu}{}^\lambda \frac{\partial}{\partial \eta^\lambda} + v_\mu \right) = \delta - ig^{\mu\nu} v_\mu \pi_\mu = e^h \delta e^{-h} \end{aligned} \quad (19)$$

where d is exterior differentiation and δ is the adjoint operator to d , that is $\delta = *d^*$ with $*$ the Hodge star operator. Thus we see that Ω and χ are the supersymmetry

operators used by Witten in his study of supersymmetry and Morse theory [1]. (The identification of states with forms also leads to a natural inner product on states; conventions for this may be found in appendix B.)

These expressions for Ω and χ simplify the calculation of the explicit expression for the canonical BRST Hamiltonian $H = -\frac{i}{2}[\Omega, \chi]$, leading to

$$H = \frac{1}{2}(d + \delta)^2 + \frac{1}{2}g^{\mu\nu} \frac{\partial h}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{i}{2}g^{\mu\lambda}(\eta^\nu \pi_\mu - \pi_\mu \eta^\nu) \frac{D^2 h}{Dx^\lambda Dx^\nu} \quad (20)$$

which is (up to a factor $\frac{1}{2}$) the Hamiltonian used by Witten [1]. Witten also shows that the mapping $\psi \mapsto e^{-h}\psi$ induces an isomorphism of de Rham cohomology classes of d and $\Omega = e^{-h}de^h$, and that forms with zero H eigenvalue give exactly one representative of each Ω cohomology class. Since additionally H has the same eigenvalues as the Laplacian $d + \delta$, we see that the gauge-fixing fermion χ is a good one [9].

4 Path Integrals

In this section stochastic calculus is used to derive a rigorous path integral expression for the action of the evolution operator $\exp -Ht$ on the states of the system.

In the special case of flat space (that is, where the manifold M is simply \mathbb{R}^n with the Euclidean metric $g^{\mu\nu} = \delta^{\mu\nu}$) this calculation was done by Salmonson and van Holten [10] using WKB methods (also known as instanton methods), which are a standard approximation technique in quantum mechanics, [11]. (An accessible account of the method applied to instantons may be found in the lectures of Coleman [12].) The basic idea is to consider fluctuations about the classical trajectories. In the conventional WKB approach only second order fluctuations are considered (first order ones vanishing because the expansion is about the classical trajectory) so that the method used is an approximate one; in this paper we give an exact path integral formula in which the usual WKB factor appears along with further factors. The approach is valid on a manifold with a general Riemannian metric as well as in flat space.

The stochastic calculus calculations which will now be given show plainly how this arises. We will begin by working in flat space, where the Hamiltonian is

$$H_f = \frac{1}{2}(d + \delta)^2 + \frac{1}{2} \frac{\partial h}{\partial x^\mu} \frac{\partial h}{\partial x^\mu} + \frac{i}{2}(\eta^\nu \pi_\mu - \pi_\mu \eta^\nu) \frac{\partial^2 h}{\partial x^\mu \partial x^\nu}. \quad (21)$$

One simple step will then adapt the method to a general Riemannian manifold.

The starting point is the stochastic differential equation

$$dx_t^\mu = db_t^\mu - v_\mu(x_t)dt, \quad x_0 = x \quad (22)$$

which implements the WKB approach of taking fluctuations about the classical trajectories, and corresponds to the Nicolai map [4]. Here x_t^μ is a stochastic process on the Wiener space of paths in \mathbb{R}^n starting from the point x and b_t^μ is a standard Brownian path in \mathbb{R}^n . Next, for positive t , we consider the operator U_t defined on functions on the superspace $\mathbb{R}^{n,n}$ by

$$U_t \psi(x, \eta) = \int d\mu \left[\exp \left(\int_0^t (\partial_\mu h(x_s) dx_s^\mu + \partial_\mu \partial_\nu h(x_s) i \theta_s^\mu \rho_{s\nu} ds) \right) \psi(x_t, \theta_t) \right] \quad (23)$$

where $d\mu$ denotes Wiener measure for paths $(b_t^\mu, \theta_s^\mu, \rho_{s\mu})$ in superspace $\mathbb{R}^{n,2n}$ (for the fermionic paths θ_s and ρ_s see [13]).

Now by Itô calculus,

$$\begin{aligned} & d \left[\exp \left(\int_0^t (\partial_\mu h(x_s) dx_s^\mu + \partial_\mu \partial_\nu h(x_s) i \theta_s^\mu \rho_{s\nu} ds) \right) \psi(x_t, \theta_t) \right] \\ &= \left[\exp \left(\int_0^t (\partial_\mu h(x_s) dx_s^\mu + \partial_\mu \partial_\nu h(x_s) i \theta_s^\mu \rho_{s\nu} ds) \right) (-H_f) \psi(x_t, \theta_t) \right] dt \\ &+ \text{terms of zero measure,} \end{aligned} \quad (24)$$

so that

$$\frac{\partial U_t f(x)}{\partial t} = -U_t H_f f(x) \quad (25)$$

and we conclude that

$$U_t = \exp -t H_f. \quad (26)$$

Now, again by Itô calculus,

$$\int_0^t \left(\partial_\mu h(x_s) dx_s^\mu + \frac{1}{2} \partial_\mu \partial_\mu h(x_s) ds \right) = h(x_t) - h(x) \quad (27)$$

so that we can simplify (23) to obtain the Feynman-Kac-Itô formula

$$\begin{aligned} \exp -t H \psi(x, \eta) &= \int d\mu \exp(-(h(x) - h(x_t))) \\ &\exp \left(\int_0^t \partial_\mu \partial_\nu h(x_s) i \theta_s^\mu \rho_{s\nu} ds \right) \psi(x_t, \theta_t). \end{aligned} \quad (28)$$

This expression shows how the WKB factor $\exp(-\Delta h)$ (which clearly corresponds to the classical action (2) of the original topological theory) appears in the path integral.

The Feynman-Kac-Itô formula is easily adapted to curved space with a general Riemannian metric $g^{\mu\nu}$ by replacing the Euclidean Brownian paths b_t with the standard Brownian paths \tilde{b}_t on a Riemannian manifold, and adjusting the fermion paths by using the (stochastic) vielbein $e_{a,s}^\mu$ as specified below. The bosonic Brownian paths on a Riemannian manifold, which were introduced by Elworthy [14] and by Ikeda and Watanabe [15], are defined by the stochastic differential equations

$$\begin{aligned} d\tilde{b}_t &= e_{a,t}^\mu db_t^a + \frac{1}{2} \Gamma_{\nu\rho}^\mu(\tilde{b}_t) dt \\ e_{a,t}^\mu &= \Gamma_{\nu\lambda}^\mu(\tilde{b}_t) d\tilde{b}_t^\nu + \frac{1}{2} e_{a,t}^\nu R_{\nu}{}^\mu(\tilde{b}_t) dt \\ \tilde{b}_0^\mu &= x^\mu, \quad e_{a,0}^\mu = e_a^\mu(x) \end{aligned} \tag{29}$$

where x is the point on the manifold from which the Brownian motion is chosen to start and $\{e_a = e_a^\mu(x) \frac{\partial}{\partial x^\mu}, a = 1 \dots n\}$ is a choice of orthonormal basis at that point. The fermionic paths $\tilde{\theta}_t^\mu, \tilde{\rho}_{t,\nu}$ are obtained from the flat space fermionic paths by rotating with the stochastic vielbein:

$$\begin{aligned} \tilde{\theta}_t^\mu &= \theta_t^a e_{a,t}^\mu \\ \tilde{\rho}_{t,\nu} &= \tilde{\rho}_{at} e_{a,t}^\mu g_{\nu\mu}(\tilde{b}_t). \end{aligned} \tag{30}$$

Using paths \tilde{x}_t on M satisfying

$$\begin{aligned} d\tilde{x}_t^\mu &= d\tilde{b}_t^\mu - g^{\nu\mu}(\tilde{x}_t) v_\nu(\tilde{x}_t) dt \\ \tilde{x}_0 &= x, \end{aligned} \tag{31}$$

similar steps to those above lead to the Feynman-Kac-Itô formula

$$\begin{aligned} \exp -tH\psi(x, \eta) &= \int d\mu \exp(-(h(x) - h(\tilde{x}_t))) \\ &\exp \left(\int_0^t (D_\mu D_\nu h(\tilde{x}_s) i g^{\lambda\nu}(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\rho}_{s\lambda} \right. \\ &\left. + R_\mu{}^\nu(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\rho}_{\nu s} + \frac{1}{2} R_{\mu\kappa}^{\lambda\nu}(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\theta}_s^\kappa \tilde{\rho}_{\lambda s} \tilde{\rho}_{\nu s}) ds \right) \psi(\tilde{x}_t, \tilde{\theta}_t). \end{aligned} \tag{32}$$

In both cases care must be taken when x is a critical point, since there will not in general be a unique solution to the stochastic differential equation concerned.

5 Morse theory and cohomology

In [1] Witten rescales the function h by a constant factor (here called u) to obtain the scaled Hamiltonian

$$H_u = \frac{1}{2}(d + \delta)^2 + u^2 g^{\mu\nu} \frac{\partial h}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} + u \frac{i}{2} g^{\mu\lambda} (\eta^\nu \pi_\mu - \pi_\mu \eta^\nu) \frac{D^2 h}{Dx^\lambda Dx^\nu}, \quad (33)$$

and, taking the large u limit, builds an explicit model for the cohomology of the manifold in terms of the critical points of the manifold with differential derived from the exterior derivative. This leads directly to the weak and strong Morse identities for M , but also gives considerably further insight into the mechanism relating the critical points to the manifold topology. Some parts of Witten's analysis have been proved rigorously, (for instance by Bismut [16] and by Simon et al [17]); however the explicit modelling of the manifold's cohomology via critical points and instanton calculations does not appear to have received a full mathematical treatment of the nature given below.

For the rest of this paper it will be assumed that h is a Morse function, that is, it has only isolated critical points. (For terminology and notation see appendix C.) If a is a critical point of h with index p then Witten [1] shows that for large u there is exactly one p -form $\psi_{(u),a}(x, \eta)$ on M which is concentrated near a and is an eigenstate of H_u with eigenvalue $\lambda_a(u)$ which is low, that is, which does not grow like u but is $o(u)$. (This result is derived analytically by Simon et al in [17].) Additionally it is shown that there are no other forms which have low H_u eigenvalues. Witten also shows (as was remarked in section 3 for the $u = 1$ case) that the mapping $\psi \mapsto e^{-hu}\psi$ induces an isomorphism of de Rham cohomology classes of d and $d_u = e^{-uh}de^{uh}$, and that forms with zero H_u eigenvalue give exactly one representative of each d_u cohomology class.

Observing that d_u and H_u commute, we see that if ψ is an eigenstate of H_u , then $d_u\psi$ is either zero or an eigenstate of H_u with the same eigenvalue. Thus if a is a critical point of h with index p ,

$$d_u\psi(a) = \sum_{b \in C_h, \text{index of } b=p+1} c_{ab}\psi_b \quad (34)$$

for some real numbers c_{ab} . As a result the cohomology of M can be modelled by p -cochains (with $p = 1, \dots, n = \dim M$) of the form

$$c = \sum_{a \in C_h, \text{index of } a=p} c_a \psi_{(u),a} \quad (35)$$

where the coefficients c_a are real numbers, with the modified exterior derivative d_u as coboundary operator. The calculation of c_{ab} in Witten's paper is done by instanton methods, which may be made both more rigorous and more transparent by using the path integral expression for $\exp -H_u t$ developed in the preceding section, as will now be described.

The constants c_{ab} in equation(34) may be evaluated by considering the matrix elements $d_{u(2)} \exp -H_u t(A, B)$ in the large u limit. (Here the notation $d_{u(2)}$ means that the operator d_u acts with respect to the second argument.) To see that these matrix elements are relevant, we choose an orthonormal basis of eigenstates of H_u consisting of the low eigenvalue states $\psi_c, c \in C_h$ (where we have simplified the notation by dropping explicit reference to u) together with further eigenstates $\{\psi_n | n = 0, \dots, \infty\}$ with eigenvalues $\lambda_n(u)$ which will be at least of order u . We can then express the matrix elements of the evolution operator as

$$\exp -H_u t(Y, X) = \sum_{c \in C_h}^{\infty} e^{-\lambda_c(u)t} {}^* \psi_c(Y) \psi_c(X) + \sum_{n=0}^{\infty} e^{-\lambda_n t} {}^* \psi_n(Y) \psi_n(X). \quad (36)$$

For large u we have an approximate expression

$$\exp -H_u t(Y, X) = \sum_{c \in C_h}^{\infty} {}^* \psi_c(Y) \psi_c(X), \quad (37)$$

so that we have at leading order for large u

$$d_{u(2)} \exp -H_u t(Y, X) = \sum_{c \in C_h}^{\infty} {}^* \psi_c(Y) d\psi_c(X). \quad (38)$$

Now as was remarked above, each ψ_c is concentrated around c . We thus expect that at leading order for large u

$$d_{u(2)} \exp -H_u t(A, B) = c_{ab} {}^* \psi_a(A) \psi_b(B), \quad (39)$$

if $\epsilon(A) = a$ and $\epsilon(B) = b$.

In order to evaluate this expression we make use of the Feynman-Kac-Itô formula (32), together with the explicit form of the kernel in the neighbourhood of a critical point. We choose a metric which globally satisfies the Smayle transversality condition for h (see appendix C), and additionally one which is Euclidean within the Morse coordinate neighbourhood N_a of each critical point a and on a neighbourhood of each steepest descent curve Γ_{ab} joining critical points.

Before proceeding further it is useful to introduce some specific coordinate systems. For each critical point c in M we will choose on N_c a fiducial set of Morse coordinates (appendix C) $x_{[c]}^\mu$ and fermionic partners $\eta_{[c]}^\mu$. Additionally for each steepest descent curve Γ_{ab} (satisfying (55)) joining the pair of critical points a and b with indices p and $p+1$ respectively we will choose a coordinate neighbourhood $U_{\Gamma_{ab}}$ which contains $N_a \cup N_b \cup U_{\Gamma_{ab}}$ with coordinates $x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}}$ such that Γ_{ab} lies along $x_{\Gamma_{ab}}^{n-p}$ while $x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}}$ match $x_{[b]}, \eta_{[b]}$ on N_b apart from possible rotations, and also match $x_{[a]}$ on N_a apart from possible rotations and (necessarily) a translation in the x^{n-p} coordinate with $x_{\Gamma_{ab}}^{n-p} = x_{[a]}^{n-p} + k_a$ for some positive constant k_a . Reconciliation with the fiducial coordinates will ultimately bring in sign factors.

Within N_a the Hamiltonian then has the form

$$H_u = \sum_{\mu=1}^n \left[\frac{1}{2} \left(-\frac{\partial^2}{\partial x_{\Gamma_{ab}}^\mu \partial x_{\Gamma_{ab}}^\mu} + u^2 (x_{\Gamma_{ab}}^\mu - a_{\Gamma_{ab}}^\mu)(x_{\Gamma_{ab}}^\mu - a_{\Gamma_{ab}}^\mu) \right) + \frac{i}{2} u \sigma_\mu (\eta_{\Gamma_{ab}}^\mu \pi_{\Gamma_{ab} \mu} - \pi_{\Gamma_{ab} \mu} \eta_{\Gamma_{ab}}^\mu) \right], \quad (40)$$

where $\sigma_\mu = 1, \mu = 1, \dots, n-p$ while $\sigma_\mu = -1, \mu = n-p+1, \dots, n$. The bosonic and fermionic parts commute so that their heat kernels may be considered separately; the bosonic part is the Harmonic oscillator Hamiltonian whose heat kernel is given by Mehler's formula [18], while the fermionic part is (apart from sign factors σ_μ) the fermionic oscillator whose heat kernel is given in [19]. If x is near a and in N_a then at leading order for large u

$$\begin{aligned} \exp -H_u t(A, X) &=_{\text{def}} M(A, X) \\ &= \left(\frac{u}{\pi} \right)^{n/2} \exp \left(-\frac{1}{2} u (x_{\Gamma_{ab}} - k_a)^2 \right) \prod_{\mu=1}^{n-p} (-\alpha_{\Gamma_{ab}}^\mu) \prod_{\nu=n-p+1}^n \eta_{\Gamma_{ab}}^\nu \end{aligned} \quad (41)$$

where X is a point over x in N_a , with coordinates $(x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}})$.

Next we calculate $\exp -H_u t(A, X)$ for x near Γ_{ab} using the Feynman-Kac-Itô formula (32). In this case the steepest descent curve (satisfying (55)) from x approaches a very fast. Thus after very small time δt the path $\tilde{x}_{\delta t}$ is almost certainly near a , so that to leading order in u we have a contribution from Γ_{ab} of

$$\begin{aligned} \exp -H_u t(A, X)_{\Gamma_{ab}} &= \exp -H_u \delta t \exp -H_u (t - \delta t)(A, X) \\ &= \int d\mu \left[\exp (-u(h(x) - h(\tilde{x}_{\delta t}))) \right] \end{aligned}$$

$$\begin{aligned}
& \times \exp \left(\int_0^{\delta t} u D_\mu D_\nu h(\tilde{x}_s) i g^{\lambda\nu}(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\rho}_{s\lambda} \right. \\
& \quad \left. + R_\mu{}^\nu(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\rho}_{\nu s} + \frac{1}{2} R_{\mu\kappa}^{\lambda\nu}(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\theta}_s^\kappa \tilde{\rho}_{\lambda s} \tilde{\rho}_{\nu s} ds \right) M(a_{\Gamma_{ab}}, \alpha_{\Gamma_{ab}}, \tilde{x}_{\delta t}, \tilde{\theta}_{\delta t}) \Big] \\
& = \left(\frac{u}{\pi} \right)^{n/2} \exp(-u(h(x) - h(a))) \prod_{\mu=1}^{n-p} (-\alpha_{\Gamma_{ab}}^\mu) \prod_{\nu=n-p+1}^n \eta_{\Gamma_{ab}}^\nu.
\end{aligned} \tag{42}$$

Here we have used the fact that the operator $\eta^\mu \pi_\mu$ which corresponds to the term $g^{\lambda\nu}(\tilde{x}_s) \tilde{\theta}_s^\mu \tilde{\rho}_{s\lambda}$ in the path integral has zero eigenvalue on $\exp -H_u t(A, \tilde{x}_{\delta t}, \tilde{\theta}_{\delta t})$ when x lies on Γ_{ab} .

To calculate $d_{u(2)} \exp -H_u t(A, B)$ we cannot take the derivative of the separate contributions from each Γ_{ab} using (42) because as we vary x around b we will jump from one Γ_{ab} to another. To avoid this difficulty we note that

$$\begin{aligned}
& d_{u(2)} \exp -H_u t(A, B) \\
& = d_{u(2)} \exp -H_u s \exp -H_u(t-s)(A, B) \\
& = \int_M d^n x d^n \eta \exp -H_u(t-s)(A, X) d_{u(2)} \exp -H_u s(X, B).
\end{aligned} \tag{43}$$

Because of the concentration of $d_{u(2)} \exp -H_u s(X, B)$ near b we can integrate over \mathbb{R}^n rather than M using the form of $\exp -H_u s(X, B)$ which is approximately true for large u on N_b ; although ultimately we will obtain a result independent of s and t , at this stage we must use Mehler's formula in full (including terms of order e^{-us} whose equivalent we could neglect near a for our purposes) because it is not the zero mode of H_u which will contribute to $d\psi_a(b)$ at leading order. Thus for x and y near b we use

$$\begin{aligned}
\exp -H_u s(X, Y) & = \left(\frac{u}{\pi} \right)^{n/2} \exp \left(-\frac{1}{2} u \left(x_{\Gamma_{ab}}^2 \frac{\cosh us}{\sinh us} \right) + u \frac{x_{\Gamma_{ab}} y_{\Gamma_{ab}}}{\sinh us} \right) \\
& \times \prod_{\mu=1}^{n-p-1} \left(\phi_{\Gamma_{ab}}^\mu e^{-us} - \eta_{\Gamma_{ab}}^\mu \right) \prod_{\nu=n-p}^n \left(\phi_{\Gamma_{ab}}^\nu - \eta_{\Gamma_{ab}}^\nu e^{-us} \right).
\end{aligned} \tag{44}$$

where $(x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}})$, $(y_{\Gamma_{ab}}, \phi_{\Gamma_{ab}})$ are the coordinates of X and Y respectively, so that at leading order in u the relevant term of $d_{u(2)} \exp -H_u s(X, B)$ (that is, the

term which contains $d\psi_a(b)$ is

$$\left(\frac{u}{\pi}\right)^{n/2} u \frac{x^{n-p}}{\sinh us} \exp\left(-\frac{1}{2}u \left(x_{\Gamma_{ab}}^2 \frac{\cosh us}{\sinh us}\right)\right) \prod_{\mu=1}^{n-p} \eta_{\Gamma_{ab}}^\mu e^{-us} \prod_{\nu=n-p}^n \beta_{\Gamma_{ab}}^\nu. \quad (45)$$

Using (43) we see that

$$\begin{aligned} & d_u(2) \exp -H_u t(A, B) \\ &= \int_{\mathbb{R}^n} d^n x_{\Gamma_{ab}} \left(\frac{u}{\pi}\right)^n \theta(x_{\Gamma_{ab}}^{n-p}) u \frac{e^{-us}}{\sinh us} x_{\Gamma_{ab}}^{n-p} \prod_{\mu=1}^{n-p} (-\alpha_{\Gamma_{ab}}^\mu) \prod_{\nu=n-p}^n \beta_{\Gamma_{ab}}^\nu \\ &\quad \times \exp\left(-\frac{1}{2}u x_{\Gamma_{ab}}^2 \frac{\cosh us}{\sinh us}\right) \exp -u(h(x) - h(a)) \\ &= \int_0^\infty dx_{\Gamma_{ab}}^{n-p} \left(\frac{u}{\pi}\right)^{(n+1)/2} u \frac{e^{-us}}{\sinh us} x_{\Gamma_{ab}}^{n-p} \prod_{\mu=1}^{n-p} (-\alpha_{\Gamma_{ab}}^\mu) \prod_{\nu=n-p}^n \beta_{\Gamma_{ab}}^\nu \\ &\quad \times \exp\left(-\frac{1}{2}u x_{\Gamma_{ab}}^2 \left(\frac{\cosh us}{\sinh us} - 1\right)\right) \exp -u(h(b) - h(a)) \\ &= \left(\frac{u}{\pi}\right)^{(n+1)/2} \prod_{\mu=1}^{n-p} (-\alpha_{\Gamma_{ab}}^\mu) \prod_{\nu=n-p}^n \beta_{\Gamma_{ab}}^\nu \exp -u(h(b) - h(a)) \\ &\quad \text{at leading order in } u. \end{aligned} \quad (46)$$

Here the θ -function occurs because the contribution from Γ_{ab} to $\exp -H_u(t-s)$ is zero on the side of b away from a . Now using equation (39) and the fact that

$$*\psi_a(A) = \left(\frac{u}{\pi}\right)^{n/4} \prod_{\mu=1}^{n-p} \alpha_{[a]}^\mu, \quad \psi_b(B) = \left(\frac{u}{\pi}\right)^{n/4} \prod_{\nu=n-p}^n \beta_{[b]}^\nu \quad (47)$$

we see that

$$c_{ab} = \left(\frac{u}{\pi}\right)^{1/2} \exp -u(h(b) - h(a)) \sum_{\Gamma_{ab}} (-1)^{\sigma_{\Gamma_{ab}}} \quad (48)$$

where $(-1)^{\sigma_{\Gamma_{ab}}}$ is a sign factor which comes from the change between the $[a]$ and $[b]$ coordinates and the $[\Gamma_{ab}]$ coordinates.

If (once again following Witten [1]) we rescale each ψ_c to $\tilde{\psi}_c = e^{-uh(c)}\psi$, and additionally use $\tilde{d}_u = \sqrt{\frac{\pi}{u}}d_u$, we obtain

$$\tilde{d}_u \tilde{\psi}_a = \sum_{\Gamma_{ab}} (-1)^{\sigma_{\Gamma_{ab}}} \tilde{\psi}_b \quad (49)$$

which coincides with the geometrical approach using ascending and descending spheres.

6 Conclusions and further possibilities

In this paper we have carried out a full canonical quantization of the simplest topological quantum theory, the topological particle, and demonstrated precisely the way in which the quantization captures topological information. The path integral formula developed in Section 4, which implements the WKB approximation in a mathematically rigorous way, even in curved space, should be useful in other situations involving quantum tunnelling.

Recent work by Hrabak [20] on the BRST operator for the two-dimensional topological sigma model leads (in an elegant and original way using the multi-symplectic formalism) to the supersymmetric theory considered and exploited by Witten [21]. A novel approach to quantization in the multi-symplectic formalism has been developed by Kanatchikov [22] which might make possible a new approach to quantization of the two dimensional model.

Appendix

A Poisson brackets

To calculate the Poisson brackets of the canonical variables x^μ , p_μ , η^μ and π_μ determined by the symplectic form (12) on the super phase space SPM introduced in section 3 we first need the Hamiltonian vector fields of these variables. The Hamiltonian vector field X_f of a function f on phase space is defined by

$$X_f \iota \omega_s = df, \quad (50)$$

where ι denotes interior product. With ω_s defined as in (12), by inspection we see that

$$\begin{aligned} X_{x^\mu} &= \frac{\partial}{\partial p_\mu} \\ X_{p_\mu} &= -\frac{\partial}{\partial x^\mu} - \Gamma_{\mu\nu}{}^\rho \pi_\rho \frac{\partial}{\partial \pi_\nu} + \Gamma_{\mu\nu}{}^\rho \eta^\nu \frac{\partial}{\partial \eta^\rho} - \frac{1}{2} R_{\mu\nu\rho}{}^\lambda \pi_\lambda \eta^\rho \frac{\partial}{\partial p_\nu} \end{aligned}$$

$$\begin{aligned}
X_{\eta^\mu} &= \frac{\partial}{\partial \pi_\mu} \\
X_{\pi_\mu} &= \frac{\partial}{\partial \eta^\mu}.
\end{aligned} \tag{51}$$

Poisson brackets are then defined by the rule

$$\{f, g\} = \frac{1}{2} (X_f \iota dg - X_g \iota df) \tag{52}$$

which leads to (14).

B Sign conventions for integrals

For the supermanifold SM Berezin integration corresponds to integration of top forms on M . If a function f on SM takes the form $f(x, \eta) = f_{\mu_1 \dots \mu_p}(x) \eta^{\mu_1} \dots \eta^{\mu_p}$ then the conventional integral

$$\int_M f_{\mu_1 \dots \mu_p}(x) dx^{\mu_1} \dots dx^{\mu_p}$$

is equal to the Berezin integral

$$\int_{SM} d^n x d^n \eta f_{\mu_1 \dots \mu_p}(x) \eta^{\mu_1} \dots \eta^{\mu_p}.$$

If K is a linear operator on functions on the supermanifold SM , then the integral kernel of K (if it exists) is defined by

$$Kf(Y) = \int_{SM} d^n x d^n \eta f(X) K(X, Y). \tag{53}$$

C Morse Theory terminology

Collected together here are some basic definitions and notation for Morse theory; more details, and many more results, may be found in the classic book of Milnor [23].

We start with a function $h : M \rightarrow \mathbb{R}$. The *critical points* of h are the points where all the partial derivatives are zero. In this paper we will assume that h is a *Morse function*, that is, its critical points are all isolated; a critical point a of h is said to be of *index* p if the Hessian matrix $(D_{x^\mu} D_{x^\nu} h(a))$ has exactly p negative eigenvalues. The set of critical points of h will be denoted C_h . (Although we give

here a coordinate-based definition of critical point and index, the definitions are of course intrinsic, independent of any choice of local coordinates.)

Each critical point a of h has a neighbourhood N_a on which special coordinate systems, known as *Morse coordinates*, can be chosen in which the Morse function h takes the standard form

$$h(x) = h(a) + \frac{1}{2} \sum_{\mu=1}^n \sigma_{\mu} (x^{\mu} - a^{\mu})^2 \quad (54)$$

with $\sigma_{\mu} = +1$ for $\mu = 1, \dots, n-p$ and $\sigma_{\mu} = -1$ for $\mu = n-p+1, \dots, n$. (Here, abusing notation for simplicity, points and their coordinates are identified.) On the corresponding neighbourhood of the supermanifold SM we use odd coordinates κ_{μ} corresponding to dx^{μ} .

A metric g on M satisfies the Smayle transversality condition for h if the solution curves Γ_{ab} to the 'steepest descent' differential equation

$$\frac{dx^{\mu}(t)}{dt} = -g^{\mu\nu} \frac{\partial h}{\partial x^{\nu}} \quad (55)$$

which start from a critical point b and end at a critical point a (with $h(a)$ necessarily less than $h(b)$) are discrete (and finite in number).

References

- [1] E. Witten. Supersymmetry and Morse theory. *Journal of Differential Geometry*, 17:661–692, 1982.
- [2] Beaulieu and I. Singer. The topological sigma model. *Commun. Math. Phys.*, 125:227–237, 1989.
- [3] H. Nicolai. On a new characterization of scalar supersymmetric theories. *Physics Letters*, B89:341–346, 1980.
- [4] D. Birmingham, M. Rakowski, and G. Thompson. Topological field theories, Nicolai maps and BRST quantization. *Physics Letters*, B214:381–386, 1988.
- [5] M. Blau, E. Keski-Vakkuri, and A.J. Niemi. Path integrals and geometry of trajectories. *Physics Letters*, B246:92–98, 1990.
- [6] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, 1992. in bfv.bib.

- [7] B. Kostant and S. Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Annals of Physics*, 176:49–113, 1987.
- [8] M. Henneaux. Hamiltonian form of the path integral for theories with a gauge freedom. *Phys. Rep.*, 126:1, 1985.
- [9] A. Rogers. Gauge fixing and BFV quantization. *Classical and Quantum Gravity*, 17:389–397, 2000.
- [10] P. Salmonson and J.W. van Holten. Fermionic coordinates and supersymmetry in quantum mechanics. *Nuclear Physics*, B196:509–531, 1982.
- [11] L.D. Landau and E.M. Lifschitz. *Quantum Mechanics*. Pergamon Press, 1958.
- [12] S. Coleman. *Aspects of Symmetry, Selected Erice Lectures*, chapter 7: The Uses of Instantons. Cambridge University Press, 1985.
- [13] Alice Rogers. Path integration, anticommuting variables and supersymmetry. *Journal of Mathematical Physics*, 36:2531–2545, 1995.
- [14] K.D. Elworthy. *Stochastic Differential Equations on Manifolds*. London Mathematical Society Lecture Notes in Mathematics. Cambridge University Press, 1982.
- [15] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. North-Holland, 1981.
- [16] J.-M. Bismut. The Witten complex and the degenerate morse inequalities. *Journal of Differential Geometry*, 23:207–240, 1986.
- [17] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrodinger operators*. Springer, 1987.
- [18] B. Simon. *Functional Integration and Quantum mechanics*. Academic Press, 1979.
- [19] A. Rogers. Fermionic path integration and Grassmann Brownian motion. *Communications in Mathematical Physics*, 113:353–368, 1987.
- [20] S.P. Hrabak. On the multisymplectic origin of the nonabelian deformation algebra of pseudoholomorphic embeddings into strictly almost kahler ambient manifolds, and the corresponding BRST algebra. Preprint math-ph/9904026, 1999.

- [21] E. Witten. Topological sigma models. *Commun.Math.Phys.*, 118:411, 1988.
- [22] I. V. Kanatchikov. On quantization of field theories in polymomentum variables. *To be published in the proceedings of International Conference on Particles, Fields and Gravitation (Devoted to the Memory of Professor Ryszard Raczka), Lodz, Poland, 15-18 Apr 1998.*, hep-th/9811016 AIP Proceedings 1998.
- [23] Milnor. *Morse Theory*. Princeton University Press, 1963.